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Comparative study of variable precision rough set model and graded rough set model

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ABSTRACT

The variable precision rough set model and graded rough set model are 2 important extended rough set models. This paper aims to make a comparative study of the 2 models. Rough set regions, primitive notions, are proposed first for the 2 models, which classify the universe more precisely. Then, both of their logical meanings related to quantitative indexes and their basic structure are investigated, and their precise descriptions are obtained as well. Furthermore, in the graded rough set model, macroscopic and microscopic algorithms are proposed and analyzed to calculate rough set regions; then, the conclusion is drawn that macroscopic and microscopic algorithms have advantages in time and space complexities, respectively, and a medical example is provided to illustrate the rough set regions and the 2 algorithms. In addition, 3 new properties of the 2 models are investigated, which are the results of extending the classical rough set model, i.e. the relationships between approximations and the basic set, the power actions of approximation operators, and the modifications of approximation operator actions on set operations. Finally, the classical rough set model is used to obtain many corresponding results, and moreover, the relationship and transformation between the 2 models is investigated. The study results of this paper have extended and enriched rough set theory from both operator-oriented and set-oriented points of view.

1. Introduction

Rough set theory was first proposed by Pawlak [14]. It is a data analysis theory and a new mathematical tool for dealing with vague and incomplete information. Studies regarding this theory may be divided into 2 kinds: set-oriented and operator-oriented studies [32]. From the set-oriented point of view, no additional operators are introduced, and classical set-theoretic operators are used to define rough set operators. Thus, rough set theory is a deviation of classical set theory. With respect to the operator-oriented view, rough set theory only introduces 2 additional operators and does not change the meanings of other set-theoretic operators; thus, it can be considered an extension of classical set theory.

The classical rough set model has a severe limitation [40]. The relationships between equivalence classes and the basic set are strict, and quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration. In fact, there are some degrees of inclusion relations between sets, and the extent of overlap of sets is important information to consider in applications. Therefore, the classical rough set model must be improved. Improved models are called generalized rough set models [17], and among them the variable precision rough set model [40] and graded rough set model [33] are 2 important ones.

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The variable precision rough set model and graded rough set model extend the classical rough set model with respect to precision and grade, respectively. Precision and grade are 2 important quantitative indexes related to relative and absolute quantization, respectively. The variable precision rough set model has already been the research focus of many theories and applications, providing many fruitful results. Refs. [2,6–8,11,24] studied reduction and rule extraction, and the model was applied in some application fields (such as medical and financial fields) in Refs. [9,12,13,16,21,26–28]. In 1990, Yao et al. proposed a decision-theoretic rough set model by using a decision-theoretic approach [35]. The variable precision rough set model is actually closely related to the decision-theoretic rough set model proposed. In [33], Yao and Lin explored the relationships between rough sets and modal logic, and proposed graded rough sets using graded modal logics and probabilistic rough sets by defining probabilistic modal logics. The probabilistic rough set model is an important model that implements the probability method, and Refs. [29–31,39] have obtained many useful results. The graded rough set model has provided some important research results, such as those reported in [34,37], though its basis, graded modal logic, has proven more fruitful [3,4,10,22,23].

The variable precision rough set model and graded rough set model can respectively reflect relative and absolute quantitative information about the degree of overlap between equivalence classes and a basic set. The relative and absolute quantitative information are 2 distinctive objective sides that describe approximate space, and each has its own virtues and application environments, so that none can be neglected. Usually, most researchers prefer using the relative quantitative information. However, the absolute quantitative information is more important than or as important as the relative quantitative information in some specific fields or special cases, and many corresponding examples can be found in practice. Here, 3 examples are provided to illustrate this point.

Example (i). You want to borrow money from 2 friends: A and B. A has 500 dollars and can lend you 400 dollars, while B has 1000 dollars and can lend you 600 dollars. If you can borrow from only one of them, who would you choose? If you only want to gain more money, B is the preferable choice, although the relative proportion of your friend's wealth is only 60%, which is lower than that (80%) of A. In this example, we specifically focus more on the absolute quantitative information and thus give low priority to the relative quantitative information, whose comparability is actually very weak.

Example (ii). University Z is the best one in a country, enrolling only the most excellent students from middle schools, and its enrollment rate is very low. There are 2 middle schools: A and B, and in a year, only 2 students from A (with 2000 students) are enrolled by University Z, while 4 students from B (with 4500 students) are admitted. Here is the question: which middle school is better in terms of enrollment? Due to the strict admission requirement, the enrollment rate is so low that the relative quantitative information (2/2000 and 4/4500) has little or not very meaningful significance. Therefore, one may simply take the absolute quantitative factor into consideration in assessments. $4 > 2$ although $4/4500 < 2/2000$, so one may conclude that B is better than A. In practice, people pay more attention to the number of enrolled students from a middle school into that university, rather than the total number of students at the middle school itself.

Example (iii). University A and University B have 40 and 20 application projects, respectively, but only 30 establishment projects are required. How does one choose between the 2? If only the relative quantitative information is considered, one can conclude that University A and University B will obtain 20 and 10 establishment projects, respectively. Is this fair in reality? If the 2 universities have almost the same research levels, then this may be reasonable. However, if the research level of University A is much higher than that of University B, then University A should obtain more than 20 establishment projects, and University B should obtain less than 10. Obviously, the number of establishment projects is the pivotal index here.

In approximate space, elements in different equivalence classes may have large differences, so different equivalence classes can reflect distinctive degrees of information. In fact, there are usually large information gaps between equivalence classes. This situation should be emphasized and utilized. However, the variable precision rough set model neglects such gaps, which can be seen in the analyses of Example (iii), possibly making it less accurate. On the contrary, the graded rough set model is concerned with the absolute quantitative information of the overlap between equivalence classes and the basic set, which can reflect the distinctive degrees of information. Therefore, it is in this situation that the graded rough set model is a key model in reflecting the quantitative information of approximate space. It is also an important supplementary model to the variable precision rough set model in certain applications, particularly those in which there are large information gaps between equivalence classes.

Therefore, the comprehensive description of relative and absolute quantitative information and the composite study of the variable precision rough set model and graded rough set model should provide more and better results. The 2 models have already been combined in [36,38], through which some initial results were obtained. As far as the current situation is concerned, the introduction of the graded rough set model into the variable precision rough set model, may improve the results obtained by the variable precision rough set model in many application fields. Therefore, the comparative study of the variable precision rough set model and graded rough set model is an innovative research field. It has both important theoretical value and wide application prospects, and it is useful for the in-depth research and application of the 2 models. Furthermore, the study is valuable to both graded modal logics and probabilistic modal logics.

Moreover, in terms of the similar basic properties of variable precision approximation operators and grade approximation operators, there is actually a close relationship between the variable precision rough set model and graded rough set model. It is against this background that this paper aims to make a comparative study of the 2 models and explore the relationship and transformation between them.

2. Preliminaries

Suppose U denotes the universe and R is an equivalence relation on U. Thus, (U, R) is called an approximate space. Suppose U/R denotes the quotient set and 2^U denotes the power set of U. Given the basic set $A \subseteq U$, if A can be denoted by some equivalence classes of R, then A is called an R definable set; otherwise, A is called an R rough set. The upper and lower approximations are defined by the following formulas: $\overline{R}A = \cup\{[x]_R : [x]_R \cap A \neq \phi\}$ and $\underline{R}A = \cup\{[x]_R : [x]_R \subseteq A\}$, where $[x]_R = \{y : xRy\}$. $\overline{R}$ and $\underline{R}$ are called upper and lower approximation operators, respectively. $posRA = \underline{R}A$, $negRA = \sim \overline{R}A$, $bnRA = \overline{R}A - \underline{R}A$ are called the R positive region, negative region, and boundary region of A, respectively [18,20].

$\overline{R}$ and $\underline{R}$ have the following properties [14]: (1) $\overline{R}\phi = \underline{R}\phi = \phi$, $\overline{R}U = \underline{R}U = U$; (2) $\underline{R}A \subseteq A \subseteq \overline{R}A$; (3) $A \subseteq B \Rightarrow \overline{R}A \subseteq \overline{R}B$, $A \subseteq B \Rightarrow \underline{R}A \subseteq \underline{R}B$; (4) $\overline{R}(A \cup B) = \overline{R}A \cup \overline{R}B$, $\underline{R}(A \cup B) \supseteq \underline{R}A \cup \underline{R}B$; (5) $\overline{R}(A \cap B) \subseteq \overline{R}A \cap \overline{R}B$, $\underline{R}(A \cap B) = \underline{R}A \cap \underline{R}B$; (6) $\overline{R}(\sim A) = \sim \underline{R}A$, $\underline{R}(\sim A) = \sim \overline{R}A$; (7) $\overline{R}(\overline{R}A) = \underline{R}(\overline{R}A) = \overline{R}A$, $\overline{R}(\underline{R}A) = \underline{R}(\underline{R}A) = \underline{R}A$.

Suppose β is in $[0, 0.5)$. β is the admissible level of classification error and is called the "error degree level" in this paper. Suppose $|A|$ denotes the cardinal number of A. $1 - \beta$ is called "precision", and $c([x]_R, A) = 1 - |[x]_R \cap A|/|[x]_R|$ is called "the relative degree of misclassification"of $[x]_R$ with respect to A. $\overline{R}_\beta A = \cup\{[x]_R : c([x]_R, A) < 1 - \beta\}$ and $\underline{R}_\beta A = \cup\{[x]_R : c([x]_R, A) \leq \beta\}$ are called precision $1 - \beta$ R upper and lower approximations of A, respectively. If $\overline{R}_\beta A = \underline{R}_\beta A$, then A is called an R definable set by precision $1 - \beta$; otherwise, A is called an R rough set by precision $1 - \beta$. $\overline{R}_\beta$ and $\underline{R}_\beta$ are called precision $1 - \beta$ upper and lower approximation operators, respectively. Suppose k is a non-negative integer and is called "grade". Thus, $\overline{R}_k A = \cup\{[x]_R : |[x]_R \cap A| > k\}$ and $\underline{R}_k A = \cup\{[x]_R : |[x]_R| - |[x]_R \cap A| \leq k\}$ are called grade k R upper and lower approximations of A, respectively. If $\overline{R}_k A = \underline{R}_k A$, then A is called an R definable set by grade k; otherwise, A is called an R rough set by grade k. $\overline{R}_k$ and $\underline{R}_k$ are called grade k upper and lower approximation operators, respectively.

$\overline{R}A$ is the union of the equivalence classes, whose elements possibly belong to A; $\underline{R}A$ is the union of the equivalence classes, whose elements necessarily belong to A. $\overline{R}_\beta A$ is the union of the equivalence classes, whose relative degree of misclassification with respect to A is less than $1 - \beta$; $\underline{R}_\beta A$ is the union of the equivalence classes, whose relative degree of misclassification with respect to A is not more than β. $\overline{R}_k A$ is the union of the equivalence classes, whose number of elements inside A is greater than k; $\underline{R}_k A$ is the union of the equivalence classes, whose number of elements outside A is not greater than k.

If $\beta = 0$ and $k = 0$, then $\overline{R}_\beta A = \overline{R}A$, $\underline{R}_\beta A = \underline{R}A$, $\overline{R}_k A = \overline{R}A$, $\underline{R}_k A = \underline{R}A$. Therefore, the classical rough set model is a special case of both the variable precision rough set model and graded rough set model. In other words, the 2 models extend the classical model.

$\overline{R}_\beta$ and $\underline{R}_\beta$ have the following properties [40]: (1) $\overline{R}_\beta\phi = \underline{R}_\beta\phi = \phi$, $\overline{R}_\beta U = \underline{R}_\beta U = U$; (2) $A \subseteq B \Rightarrow \overline{R}_\beta A \subseteq \overline{R}_\beta B$, $A \subseteq B \Rightarrow \underline{R}_\beta A \subseteq \underline{R}_\beta B$; (3) $\overline{R}_\beta(A \cup B) \supseteq \overline{R}_\beta A \cup \overline{R}_\beta B$, $\underline{R}_\beta(A \cup B) \supseteq \underline{R}_\beta A \cup \underline{R}_\beta B$; (4) $\overline{R}_\beta(A \cap B) \subseteq \overline{R}_\beta A \cap \overline{R}_\beta B$, $\underline{R}_\beta(A \cap B) \subseteq \underline{R}_\beta A \cap \underline{R}_\beta B$; (5) $\overline{R}_\beta(\sim A) = \sim \underline{R}_\beta A$, $\underline{R}_\beta(\sim A) = \sim \overline{R}_\beta A$; (6) $\beta \geq \alpha \Leftrightarrow \overline{R}_\beta A \subseteq \overline{R}_\alpha A$, $\underline{R}_\beta A \supseteq \underline{R}_\alpha A$; (7) $\overline{R}_0 A = \overline{R}A$, $\underline{R}_0 A = \underline{R}A$.

Similarly, $\overline{R}_k$ and $\underline{R}_k$ have the following properties [33]: (1) $\overline{R}_k\phi = \phi$, $\underline{R}_k\phi = \cup\{[x]_R : |[x]_R| \leq k\}$, $\overline{R}_k U = \cup\{[x]_R : |[x]_R| > k\}$, $\underline{R}_k U = U$; (2) $A \subseteq B \Rightarrow \overline{R}_k A \subseteq \overline{R}_k B$, $A \subseteq B \Rightarrow \underline{R}_k A \subseteq \underline{R}_k B$; (3) $\overline{R}_k(A \cup B) \supseteq \overline{R}_k A \cup \overline{R}_k B$, $\underline{R}_k(A \cup B) \supseteq \underline{R}_k A \cup \underline{R}_k B$; (4) $\overline{R}_k(A \cap B) \subseteq \overline{R}_k A \cap \overline{R}_k B$, $\underline{R}_k(A \cap B) \subseteq \underline{R}_k A \cap \underline{R}_k B$; (5) $\overline{R}_k(\sim A) = \sim \underline{R}_k A$, $\underline{R}_k(\sim A) = \sim \overline{R}_k A$; (6) $k \geq l \Leftrightarrow \overline{R}_k A \subseteq \overline{R}_l A$, $\underline{R}_k A \supseteq \underline{R}_l A$; (7) $\overline{R}_0 A = \overline{R}A$, $\underline{R}_0 A = \underline{R}A$.

Usually, β is in $[0, 0.5)$ in the variable precision rough set model [40]. However, An et al. [1] and Beynon [2] used the symbol β to denote the proportion of correct classifications, in which case the appropriate range is $(0.5, 1]$; this definition of β was also used in reference [7]. They referred to this technique as enhanced rough set theory. In this paper, for completeness, the range of parameter β is improved and expanded to $[0, 1]$. The mentioned properties of the variable precision rough set model are the basic ones; moreover, the in-depth studies and properties of the model are investigated in [5,8,19,21]. Rough membership is an important measure in the model, which is introduced by Wong and Ziarko [25], and its properties are extensively investigated in [5,15,29].

3. Graded rough set model and its properties

3.1. Rough set regions and their algorithms in the graded rough set model

Definition 1. $posR_k A = \overline{R}_k A \cap \underline{R}_k A$, $negR_k A = \sim (\overline{R}_k A \cup \underline{R}_k A)$, $UbnR_k A = \overline{R}_k A - \underline{R}_k A$, $LbnR_k A = \underline{R}_k A - \overline{R}_k A$, $bnR_k A = UbnR_k A \cup LbnR_k A$ are called the grade k R positive region, negative region, upper boundary region, lower boundary region, and boundary region of A, respectively. These new notions, as well as the grade upper and lower approximations, are all called the rough set regions of the graded rough set model.

The rough set regions of the graded rough set model are the extensions of grade approximations. Because the inclusion relation of the grade approximations does not hold any longer, positive and negative regions, upper and lower boundary regions are naturally proposed. Obviously, rough set regions of the graded rough set model also extend the corresponding notions of the classical rough set model. More importantly, they classify the universe more precisely and have their own logical meanings related to the grade quantitative index.

The positive region $posR_kA$ is the union of the equivalence classes, whose number of elements inside A is greater than k and whose number of elements outside A is not greater than k; the negative region $negR_kA$ is the union of the equivalence classes, whose number of elements inside A is not greater than k and whose number of elements outside A is greater than k; the upper boundary region $UbnR_kA$ is the union of the equivalence classes, whose number of elements inside (outside) A is greater than k; the lower boundary region $LbnR_kA$ is the union of the equivalence classes, whose number of elements inside (outside) A is not greater than k; the boundary region bnR_kA is the union of the equivalence classes, whose number of elements inside (outside) A is greater than k or not greater than k.

Proposition 1

$$\overline{R}_kA = posR_kA \cup UbnR_kA,$$
$$\underline{R}_kA = posR_kA \cup LbnR_kA.$$

Definition 1 and Proposition 1 present the basic structure of the rough set regions of the graded rough set model. Obviously, the grade upper and lower approximations are the core notions, which can be used to calculate other rough set regions. On the other hand, the positive and negative regions, upper and lower boundary regions constitute a partition of the universe, which can calculate the rest as well. These basic properties of rough set regions are just the bases of macroscopic and microscopic algorithms, which will be proposed and analyzed below.

Proposition 2. *When $k = 0$,*

$$posR_kA = \underline{R}A, \quad negR_kA = \sim \overline{R}A, \quad UbnR_kA = \overline{R}A - \underline{R}A, \quad LbnR_kA = \phi.$$

Theorem 1. *When $k \neq 0$,*

(1) $posR_kA = (\cup\{[x]_R : |[x]_R| > 2k, |[x]_R \cap A| \geq |[x]_R| - k\}) \cup (\cup\{[x]_R : |[x]_R| \in (k, 2k], |[x]_R \cap A| > k\})$;
(2) $negR_kA = (\cup\{[x]_R : |[x]_R| > 2k, |[x]_R \cap A| \leq k\}) \cup (\cup\{[x]_R : |[x]_R| \in (k, 2k], |[x]_R \cap A| < |[x]_R| - k\})$;
(3) $UbnR_kA = \cup\{[x]_R : |[x]_R| > 2k, |[x]_R \cap A| \in (k, |[x]_R| - k)\}$;
(4) $LbnR_kA = (\cup\{[x]_R : |[x]_R| \in (k, 2k], |[x]_R \cap A| \in [|[x]_R| - k, k]\}) \cup (\cup\{[x]_R : |[x]_R| \in [1, k]\})$.

Proof

(1) If $|[x]_R| > 2k$, then $|[x]_R| - k > k$. Hence, $|[x]_R \cap A| \geq |[x]_R| - k \Rightarrow |[x]_R \cap A| > k$. From the definitions, $[x]_R \subseteq \underline{R}_kA \Rightarrow [x]_R \subseteq \overline{R}_kA$. (i) If $|[x]_R \cap A| \geq |[x]_R| - k$, then $[x]_R \subseteq \underline{R}_kA$ and $[x]_R \subseteq posR_kA$; (ii) If $|[x]_R \cap A| \leq k$, then $[x]_R \subseteq U - \overline{R}_kA$ and $[x]_R \subseteq negR_kA$; (iii) If $|[x]_R \cap A| \in (k, |[x]_R| - k)$, then $[x]_R \subseteq \overline{R}_kA - \underline{R}_kA = UbnR_kA$.
(2) If $|[x]_R| \in (k, 2k]$, then $0 < |[x]_R| - k \leq k$. Hence, $|[x]_R \cap A| > k \Rightarrow |[x]_R \cap A| \geq |[x]_R| - k$. From the definitions, $[x]_R \subseteq \overline{R}_kA \Rightarrow [x]_R \subseteq \underline{R}_kA$. (i) If $|[x]_R \cap A| > k$, then $[x]_R \subseteq \overline{R}_kA$ and $[x]_R \subseteq posR_kA$; (ii) If $|[x]_R \cap A| < |[x]_R| - k$, then $[x]_R \subseteq U - \underline{R}_kA$ and $[x]_R \subseteq negR_kA$; (iii) If $|[x]_R \cap A| \in [|[x]_R| - k, k]$, then $[x]_R \subseteq \underline{R}_kA - \overline{R}_kA = LbnR_kA$.
(3) If $|[x]_R| \in [1, k]$, then $|[x]_R| - k \leq 0$ and $0 \leq |[x]_R \cap A| \leq k$. $|[x]_R \cap A| \geq 0 \geq |[x]_R| - k$, so $[x]_R \subseteq \underline{R}_kA$. $|[x]_R \cap A| \leq k$, so $[x]_R \not\subseteq \overline{R}_kA$. Hence, $[x]_R \subseteq \underline{R}_kA - \overline{R}_kA = LbnR_kA$.

Based on the results of (1)–(3) as well as the completeness of the discussions, the results of the theorem can be achieved naturally and easily. □

The case $k = 0$ is very simple, and Proposition 2 provides the results of the rough set regions of the graded rough set model. $[x]_R \in U/R \Rightarrow |[x]_R| \geq 1 > 0$. Therefore, the classical rough set model (where $k = 0$) is only a special case of the proof of Theorem 1. Theorem 1 provides the precise formulas used to calculate the positive and negative regions, upper and lower boundary regions in the general case. Furthermore, all rough set regions of the graded rough set model can be described precisely. There are only 3 types of relationships between $|[x]_R|$ and k, $2k$, and there are also only 3 types of relationships between $|[x]_R \cap A|$ and k, $|[x]_R| - k$. If $[x]_R \in [1, k]$, then undoubtedly $[x]_R \subseteq LbnR_kA$. Therefore, there are only 7 cases of the distributions of equivalence classes. Table 1 shows the detailed distributions of equivalence classes as well as their complete attributions with respect to the basic rough set regions.

Table 1
Distributions, attributions and microscopic algorithm analyses of equivalence classes.

Case $\|[x]_R\|$	$\|[x]_R \cap A\|$	Rough set region	Number of comparisons	Number of auxiliary variables
(1) $(2k, +\infty)$	$[0, k]$	$negR_kA$	2	0
(2) $(2k, +\infty)$	$(k, \|[x]_R\| - k)$	$UbnR_kA$	3	1
(3) $(2k, +\infty)$	$[\|[x]_R\| - k, +\infty)$	$posR_kA$	3	1
(4) $(k, 2k]$	$(k, +\infty)$	$posR_kA$	3	0
(5) $(k, 2k]$	$[\|[x]_R\| - k, k]$	$LbnR_kA$	4	1
(6) $(k, 2k]$	$[0, \|[x]_R\| - k)$	$negR_kA$	4	1
(7) $[1, k]$	–	$LbnR_kA$	2	0

Now, in the general case that k is a positive integer, 2 algorithms will be proposed and analyzed to calculate the rough set regions of the graded rough set model: macroscopic and microscopic algorithms.

Algorithm 1 (Macroscopic algorithm)

Step 1. Calculate grade upper and lower approximations using definitions.
Step 2. Obtain positive and negative regions, upper and lower boundary regions, and the overall boundary region using set operations (Definition 1).

Algorithm 2 (Microscopic algorithm)

Step 1. Calculate positive and negative regions, upper and lower boundary regions using Theorem 1;
Step 2. Obtain grade upper and lower approximations and boundary region using union operation of sets and Proposition 1.

The calculation process of the macroscopic algorithm is very clear. As for the determination property, more precise descriptions of the calculation process of the microscopic algorithm, especially the comparison order of the parameters, will be given. Here, $|[x]_R|$ is compared with $2k$ first and k second in the first comparison process, while $|[x]_R \cap A|$ is compared with k first and $|[x]_R| - k$ second in the second process.

The core task of the 2 algorithms is to determine whether each equivalence class belongs to a specific set, where the main calculation is the comparison. For each equivalence class, 2 input data, $|[x]_R|$ and $|[x]_R \cap A|$, are required. Suppose there are n equivalence classes; thus, there are $2n$ input data. Now, the comparison will be chosen as the basic operation to analyze and compare the 2 algorithms.

In the macroscopic algorithm, each equivalence class requires calculation to show whether it belongs to $\overline{R}_k A$ or $\underline{R}_k A$ first, and it needs to be compared twice using 1 auxiliary variable: $|[x]_R| - |[x]_R \cap A|$. Thus, the other processes are set operations. Hence, time and space complexities are as follows: $T(n) = 2n$ and $S(n) = n$, and the results are invariable.

In the microscopic algorithm, there are only 7 cases of equivalence classes. Table 1 also shows the analyses of number of comparisons and number of auxiliary variables required in each case. In the worst case, the time and space complexities are as follows: $T(n) = 4n$ and $S(n) = n$; in the best case, the time and space complexities are as follows: $T(n) = 2n$ and $S(n) = c$.

The macroscopic algorithm is based on the following core notions of the grade upper and lower approximations, while the microscopic algorithm is based on the basic rough set regions of the partition: the positive and negative regions, upper and lower boundary regions. In the worst case, the asymptotic analyses of the time and space complexities of the 2 algorithms are the same: $T(n) = \Theta(n)$ and $S(n) = \Theta(n)$. However, the macroscopic algorithm has an advantage with respect to the time complexity, and the time complexity of the macroscopic algorithm is the lower bound of that of the microscopic algorithm. On the other hand, the microscopic algorithm has an advantage with respect to the space complexity, and the space complexity of the macroscopic algorithm is the upper bound of that of the microscopic algorithm. Therefore, the macroscopic and microscopic algorithms have advantages with respect to time and space complexities, respectively. Why is this? In actuality, the calculations of the grade upper and lower approximations are the simple basic operations in the macroscopic algorithm. On the other hand, in the microscopic algorithm, the basic rough set regions are described precisely, so more auxiliary variables are not required in some cases, such as Case (1), (4) and (7) in Table 1.

In practice, the macroscopic or microscopic algorithm may be chosen appropriately according to the concrete time or space complexities requirements. In the first comparison process of the microscopic algorithm, the order of parameters compared with $|[x]_R|$ is such that $2k$ is followed by k. This can be applied to the case in which either k is slightly smaller or in which the number of equivalence classes with high cardinal numbers is slightly greater. Otherwise, the other order may be chosen, i.e., k is followed by $2k$, when the microscopic algorithm can be analyzed similarly. When processing massive amounts of data, the comparison order of parameters may be chosen by ordering $|[x]_R|$ and locating the relationships between $|[x]_R|$ and k. Meanwhile, in the second comparison process of the microscopic algorithm, the order of parameters compared with $|[x]_R \cap A|$ is such that k is followed by $|[x]_R| - k$. This order can reduce the number of auxiliary variables and thus decrease the space complexity of the microscopic algorithm.

3.2. A medical example

Example 1. $S = (U, T, V, f)$ is a decision table. U is composed of 36 patients, and $T = \{r_1, r_2, r_3\}$. Condition attribute r_1 and r_2 represent “fever” and “headache”, respectively, and decision attribute r_3 represents “cold”. R denotes the equivalence relation based on r_1 and r_2. $V_{r_1} = \{0, 1, 2\}$, where 0, 1 and 2 represent “no fever”, “mild fever”and “severe fever”, respectively. $V_{r_2} = \{0, 1, 2\}$, where 0, 1 and 2 represent “no headache”, “mild headache” and “severe headache”, respectively. $V_{r_3} = \{0, 1\}$, where 0 and 1 represent “no cold” and “cold”, respectively. Based on the initial medical data (Table 2), the statistical results of the patient classes (Table 3) can be obtained, where $[x]_m = (i, j)(m = 1, 2, \ldots, 9)$ denote the patient classes based on “fever” and “headache” condition attributes, and A denotes the cold patient set.

Table 2
Initial medical data.

Patient	Fever	Headache	Cold	Patient	Fever	Headache	Cold
1	0	0	0	19	0	0	0
2	1	1	0	20	1	2	1
3	0	2	1	21	2	0	1
4	2	1	0	22	0	0	0
5	1	0	1	23	2	1	0
6	2	2	1	24	1	2	1
7	0	0	0	25	0	2	0
8	1	2	0	26	2	2	1
9	2	2	1	27	1	1	0
10	1	1	1	28	2	0	1
11	1	2	1	29	2	1	1
12	2	0	0	30	0	0	0
13	0	0	0	31	1	2	0
14	2	1	1	32	0	1	0
15	0	1	1	33	2	1	1
16	1	1	0	34	1	1	1
17	0	2	0	35	0	0	0
18	2	1	1	36	2	0	0

Table 3
Statistical results of the patient classes.

$[x]_m$	Elements of $[x]_m$	$\lvert[x]_m\rvert$	Elements of $[x]_m \cap A$	$\lvert[x]_m \cap A\rvert$	$c([x]_m, A)$	$\lvert[x]_m\rvert - \lvert[x]_m \cap A\rvert$
$[x]_1 = (0, 0)$	1, 7, 13, 19, 22, 30, 35	7	–	0	1	7
$[x]_2 = (0, 1)$	15, 32	2	15	1	0.5	1
$[x]_3 = (0, 2)$	3, 17, 25	3	3	1	2/3	2
$[x]_4 = (1, 0)$	5	1	5	1	0	0
$[x]_5 = (1, 1)$	2, 10, 16, 27, 34	5	10, 34	2	0.6	3
$[x]_6 = (1, 2)$	8, 11, 20, 24, 31	5	11, 20, 24	3	0.4	2
$[x]_7 = (2, 0)$	12, 21, 28, 36	4	21, 28	2	0.5	2
$[x]_8 = (2, 1)$	4, 14, 18, 23, 29, 33	6	14, 18, 29, 33	4	1/3	2
$[x]_9 = (2, 2)$	6, 9, 26	3	6, 9, 26	3	0	0

Table 4
Calculations and analyses of microscopic algorithm of the medical example.

Patient class: $[x]_m$	Case	The first comparison process: $\lvert[x]_m\rvert$	The second comparison process: $\lvert[x]_m \cap A\rvert$	Rough set region	Number of comparisons	Number of auxiliary variables
$[x]_1$	(1)	$(2, +\infty)$	$[0, 1]$	$negR_1A$	2	0
$[x]_2$	(5)	$(1, 2]$	$[\lvert[x]_R\rvert - 1, 1]$	$LbnR_1A$	4	1
$[x]_3$	(1)	$(2, +\infty)$	$[0, 1]$	$negR_1A$	2	0
$[x]_4$	(7)	$[1, 1]$	–	$LbnR_1A$	2	0
$[x]_5$	(2)	$(2, +\infty)$	$(1, \lvert[x]_R\rvert - 1)$	$UbnR_1A$	3	1
$[x]_6$	(2)	$(2, +\infty)$	$(1, \lvert[x]_R\rvert - 1)$	$UbnR_1A$	3	1
$[x]_7$	(2)	$(2, +\infty)$	$(1, \lvert[x]_R\rvert - 1)$	$UbnR_1A$	3	1
$[x]_8$	(2)	$(2, +\infty)$	$(1, \lvert[x]_R\rvert - 1)$	$UbnR_1A$	3	1
$[x]_9$	(3)	$(2, +\infty)$	$[\lvert[x]_R\rvert - 1, +\infty)$	$posR_1A$	3	1

The rough set regions of the graded rough set model will be calculated in the case that $k = 1$.

Macroscopic algorithm. Step 1. $\overline{R}_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_1A = [x]_2 \cup [x]_4 \cup [x]_9$; Step 2. $posR_1A = [x]_9$, $negR_1A = [x]_1 \cup [x]_3$, $UbnR_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$, $LbnR_1A = [x]_2 \cup [x]_4$, $bnR_1A = [x]_2 \cup [x]_4 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$.

Microscopic algorithm. Step 1. By calculation, the following results can be obtained: $[x]_9 \subseteq posR_1A$; $[x]_1, [x]_3 \subseteq negR_1A$; $[x]_5, [x]_6, [x]_7, [x]_8 \subseteq UbnR_1A$; $[x]_2, [x]_4 \subseteq LbnR_1A$. The detailed calculations are shown in Table 4. Furthermore, it can be obtained that: $posR_1A = [x]_9$, $negR_1A = [x]_1 \cup [x]_3$, $UbnR_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$, $LbnR_1A = [x]_2 \cup [x]_4$; Step 2. $\overline{R}_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_1A = [x]_2 \cup [x]_4 \cup [x]_9$, $bnR_1A = [x]_2 \cup [x]_4 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$.

According to "fever" and "headache", the universe is divided into 9 patient classes. $\overline{R}A = [x]_2 \cup [x]_3 \cup [x]_4 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$ and $\underline{R}A = [x]_4 \cup [x]_9$ denote these patient classes, whose elements possibly and necessarily belong to the cold patient set, respectively.

$\overline{R}_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$ denotes these patient classes, whose number of elements inside the cold patient set is greater than 1; $\underline{R}_1A = [x]_2 \cup [x]_4 \cup [x]_9$ denotes these patient classes, whose number of elements outside the cold patient set is not greater than 1; $posR_1A = [x]_9$ denotes these patient classes, whose number of elements inside the cold patient set is greater than 1 and whose number of elements outside the cold patient set is not greater than 1; $negR_1A = [x]_1 \cup [x]_3$ denotes these patient classes, whose number of elements inside the cold patient set is not greater than 1 and whose number of elements outside the cold patient set is greater than 1; $UbnR_1A = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$

denotes these patient classes, whose number of elements inside (outside) the cold patient set is greater than 1; $LbnR_1A = [x]_2 \cup [x]_4$ denotes these patient classes, whose number of elements inside (outside) the cold patient set is not greater than 1; $bnR_1A = [x]_2 \cup [x]_4 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$ denotes these patient classes, whose number of elements inside (outside) the cold patient set is greater than 1 or not greater than 1.

Example 1 shows that the rough set regions of the graded rough set model make a composite description with respect to the grade quantitative index and have practical significance. There are only 9 patient classes in this example. The time and space complexities of the macroscopic algorithm are as follows: $T(9) = 18$ and $S(9) = 9$, while those of the microscopic algorithm are as follows: $T(9) = 25$ and $S(9) = 6$. The detailed analyses of the microscopic algorithm are shown in Table 4. Both the time advantage of the macroscopic algorithm and the space advantage of the microscopic algorithm are very clear.

3.3. Properties of the graded rough set model

The graded rough set model is an extension of the classical rough set model. Unlike the classical one, however, it lacks the following properties: (1) $\underline{R}A \subseteq A \subseteq \overline{R}A$, (2) $\overline{R}(\overline{R}A) = \underline{R}(\overline{R}A) = \overline{R}A$, $\overline{R}(\underline{R}A) = \underline{R}(\underline{R}A) = \underline{R}A$. Meanwhile, the equality formulas have been turned into $\overline{R}_k(A \cup B) \supseteq \overline{R}_kA \cup \overline{R}_kB$ and $\underline{R}_k(A \cap B) \subseteq \underline{R}_kA \cap \underline{R}_kB$. Here, the new properties of the graded rough set model will be studied with respect to the 3 properties, which are the results of extending the classical rough set model.

Proposition 3

$$LbnR_kA = \phi \Leftrightarrow \underline{R}_kA \subseteq \overline{R}_kA,$$
$$UbnR_kA = \phi \Leftrightarrow \overline{R}_kA \subseteq \underline{R}_kA,$$
$$LbnR_kA = UbnR_kA = \phi \Leftrightarrow bnR_kA = \phi \Leftrightarrow \underline{R}_kA = \overline{R}_kA.$$

Generally speaking, $\underline{R}_kA$ and $\overline{R}_kA$ do not have an inclusion relation or equality relation. In practice, "evaluation mappings of sets"may be introduced to comprehensively estimate sets.

Definition 2. Suppose I is a commutative group, $+$ is the operator, and 0 is the identity element. $\forall a \in I$, the inverse element of a is denoted as $-a$. Suppose $\leq$ is a total order relation on I and $A, B \subseteq U$. Now, f is a mapping from 2^U to I and satisfies the following conditions: (1) $f(\phi) = 0$; (2) $A \subseteq B \Rightarrow f(A) \leq f(B)$; (3) $f(A \cup B) = f(A) + f(B) - f(A \cap B)$; (4) $f(A - B) = f(A) - f(A \cap B)$, and hence, $A = U \Rightarrow f(\sim B) = f(U) - f(B)$, $A \cap B = \phi \Rightarrow f(A - B) = f(A)$. Then, f is called an I evaluation mapping of U.

Example 2. Suppose $I = Z$ (integer set), + is the conventional addition operation, 0 is the conventional number 0, and $\leq$ is the "no greater than" relation. Let $f(A) = |A|$; thus, f is a Z evaluation mapping of U.

Proposition 4

$$f(LbnR_kA) \leq f(UbnR_kA) \Longleftrightarrow f(\underline{R}_kA) \leq f(\overline{R}_kA),$$
$$f(UbnR_kA) \leq f(LbnR_kA) \Longleftrightarrow f(\overline{R}_kA) \leq f(\underline{R}_kA),$$
$$f(LbnR_kA) = f(UbnR_kA) \Longleftrightarrow f(\underline{R}_kA) = f(\overline{R}_kA).$$

Proposition 4 estimates the relationship between $\overline{R}_kA$ and $\underline{R}_kA$. Furthermore, the relationships between them and A may be estimated similarly based on A.

Theorem 2

(1) $\overline{R}_k(\overline{R}_kA) = \overline{R}_kA \subseteq \underline{R}_k(\overline{R}_kA)$;
(2) $\underline{R}_k(\overline{R}_kA) = \overline{R}_kA \cup (\cup\{[x]_R : |[x]_R| \leq k\})$;
(3) $\overline{R}_k(\underline{R}_kA) \subseteq \underline{R}_kA = \underline{R}_k(\underline{R}_kA)$;
(4) $\underline{R}_kA = \overline{R}_k(\underline{R}_kA) \cup (\cup\{[x]_R : |[x]_R| \leq k\})$.

Proof

(1) If $[x]_R \subseteq \overline{R}_k(\overline{R}_kA)$, then $|[x]_R \cap \overline{R}_kA| > k$. If $[x]_R \nsubseteq \overline{R}_kA$, then $|[x]_R \cap \overline{R}_kA| = 0$. This contradicts the result. Hence, $[x]_R \subseteq \overline{R}_kA$, and $\overline{R}_k(\overline{R}_kA) \subseteq \overline{R}_kA$. If $[x]_R \subseteq \overline{R}_kA$, then $|[x]_R \cap A| > k$; therefore, $|[x]_R \cap \overline{R}_kA| = |[x]_R| \geq |[x]_R| - k$

and $|[x]_R \cap \overline{R}_k A| = |[x]_R| \geq |[x]_R \cap A| > k$. Hence, $[x]_R \subseteq \underline{R}_k(\overline{R}_k A)$ and $[x]_R \subseteq \overline{R}_k(\overline{R}_k A)$. Hence, $\overline{R}_k A \subseteq \underline{R}_k(\overline{R}_k A)$ and $\overline{R}_k A \subseteq \overline{R}_k(\overline{R}_k A)$.

(2) If $|[x]_R| \leq k$ then $|[x]_R| - k \leq 0$, so $|[x]_R \cap \overline{R}_k A| \geq |[x]_R| - k$, i.e., $[x]_R \subseteq \underline{R}_k(\overline{R}_k A)$. From (1) $\overline{R}_k A \subseteq \underline{R}_k(\overline{R}_k A)$, so $\underline{R}_k(\overline{R}_k A) \supseteq \overline{R}_k A \cup (\cup\{[x]_R : |[x]_R| \leq k\})$. On the contrary, if $[x]_R \subseteq \underline{R}_k(\overline{R}_k A)$, then $|[x]_R \cap \overline{R}_k A| \geq |[x]_R| - k$. If $|[x]_R| - k > 0$ (i.e. $|[x]_R| > k$), then $[x]_R \subseteq \overline{R}_k A$ and $|[x]_R \cap \overline{R}_k A| \geq |[x]_R| - k$. If $|[x]_R| - k \leq 0$ (i.e., $|[x]_R| \leq k$), then $[x]_R \nsubseteq \overline{R}_k A$ and $[x]_R \subseteq \cup\{[x]_R : |[x]_R| \leq k\}$; hence, $\underline{R}_k(\overline{R}_k A) \subseteq \overline{R}_k A \cup (\cup\{[x]_R : |[x]_R| \leq k\})$. Hence, $\underline{R}_k(\overline{R}_k A) = \overline{R}_k A \cup (\cup\{[x]_R : |[x]_R| \leq k\})$.

(3) If $[x]_R \subseteq \overline{R}_k(\underline{R}_k A)$, then $|[x]_R \cap \underline{R}_k A| > k$. If $[x]_R \nsubseteq \underline{R}_k A$, then $|[x]_R \cap \underline{R}_k A| = 0$. This contradicts the result. Hence, $[x]_R \subseteq \underline{R}_k A$, and $\overline{R}_k(\underline{R}_k A) \subseteq \underline{R}_k A$. If $[x]_R \subseteq \underline{R}_k A$, then $|[x]_R \cap A| \geq |[x]_R| - k$. Then, $|[x]_R \cap \underline{R}_k A| = |[x]_R| \geq |[x]_R| - k$, so $[x]_R \subseteq \underline{R}_k(\underline{R}_k A)$ and $\underline{R}_k A \subseteq \underline{R}_k(\underline{R}_k A)$. On the contrary, if $[x]_R \subseteq \underline{R}_k(\underline{R}_k A)$, then $|[x]_R \cap \underline{R}_k A| \geq |[x]_R| - k$. If $[x]_R \nsubseteq \underline{R}_k A$, then $|[x]_R \cap \underline{R}_k A| = 0$ and $0 \leq |[x]_R \cap A| < |[x]_R| - k$. This contradicts $0 \geq |[x]_R| - k$, so $[x]_R \subseteq \underline{R}_k A$ and $\underline{R}_k(\underline{R}_k A) \subseteq \underline{R}_k A$. Hence, $\underline{R}_k A = \underline{R}_k(\underline{R}_k A)$.

(4) If $|[x]_R| \leq k$, then $[x]_R \subseteq \underline{R}_k A$. From (3), $\overline{R}_k(\underline{R}_k A) \subseteq \underline{R}_k A$, so $\underline{R}_k A \supseteq \overline{R}_k(\underline{R}_k A) \cup (\cup\{[x]_R : |[x]_R| \leq k\})$. On the contrary, if $[x]_R \subseteq \underline{R}_k A$, then $|[x]_R \cap \underline{R}_k A| = |[x]_R|$. If $|[x]_R| > k$, then $[x]_R \subseteq \overline{R}_k(\underline{R}_k A)$. Otherwise, $|[x]_R| \leq k$, so $\underline{R}_k A \subseteq \overline{R}_k(\underline{R}_k A) \cup (\cup\{[x]_R : |[x]_R| \leq k\})$. Hence, $\underline{R}_k A = \overline{R}_k(\underline{R}_k A) \cup (\cup\{[x]_R : |[x]_R| \leq k\})$. $\square$

According to Theorem 2, the grade approximation operators have the idempotent properties when the composite directions are the same (i.e., grade upper approximation operator or grade lower approximation operator make the composition). On the contrary, when the grade upper approximation operator and grade lower approximation operator make the composition, the idempotent properties no longer hold. In the composite process, the grade upper approximation operator will reduce its acting grade approximation set, while the grade lower approximation operator will enlarge its acting one. From the proof of Theorem 1, if $|[x]_R| \in [1, k]$, then $[x]_R \subseteq \underline{R}_k A$ and $[x]_R \nsubseteq \overline{R}_k A$. It is just this special property that hinders the idempotent properties of the grade approximation operators.

Definition 3

$$bnR_k\overline{I}(A, B) = \cup\{[x]_R : [x]_R \nsubseteq \overline{R}_k A, \overline{R}_k B, [x]_R \subseteq \overline{R}_k(A \cup B)\},$$

$$bnR_k\underline{I}(A, B) = \cup\{[x]_R : [x]_R \nsubseteq \underline{R}_k A, \underline{R}_k B, [x]_R \subseteq \underline{R}_k(A \cup B)\},$$

$$bnR_k\overline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq \overline{R}_k A, \overline{R}_k B, [x]_R \nsubseteq \overline{R}_k(A \cap B)\},$$

$$bnR_k\underline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq \underline{R}_k A, \underline{R}_k B, [x]_R \nsubseteq \underline{R}_k(A \cap B)\}$$

are called the grade k R upper inner, lower inner, upper outer, and lower outer boundary regions of A and B, respectively. $bnR_k\overline{I}$, $bnR_k\underline{I}$, $bnR_k\overline{O}$ and $bnR_k\underline{O}$ are called the grade k upper inner, lower inner, upper outer, and lower outer boundary operators, respectively.

Definition 4. New union, intersection and complement operations of grade approximations (noted as $\underline{\cup}$, $\overline{\cap}$, $\sim^*$) are defined as follows:

(1) $\overline{R}_k A \underline{\cup} \overline{R}_k B = \overline{R}_k A \cup \overline{R}_k B \cup bnR_k\overline{I}(A, B)$,
$\underline{R}_k A \underline{\cup}\, \underline{R}_k B = \underline{R}_k A \cup \underline{R}_k B \cup bnR_k\underline{I}(A, B)$;
(2) $\overline{R}_k A \overline{\cap}\, \overline{R}_k B = \overline{R}_k A \cap \overline{R}_k B - bnR_k\overline{O}(A, B)$,
$\underline{R}_k A \overline{\cap} \underline{R}_k B = \underline{R}_k A \cap \underline{R}_k B - bnR_k\underline{O}(A, B)$;
(3) $\sim^* \overline{R}_k A = \sim \underline{R}_k A$, $\sim^* \underline{R}_k A = \sim \overline{R}_k A$.

Proposition 5

(1) $\overline{R}_k(A \cup B) = \overline{R}_k A \underline{\cup} \overline{R}_k B$, $\underline{R}_k(A \cup B) = \underline{R}_k A \underline{\cup}\, \underline{R}_k B$;
(2) $\overline{R}_k(A \cap B) = \overline{R}_k A \overline{\cap}\, \overline{R}_k B$, $\underline{R}_k(A \cap B) = \underline{R}_k A \overline{\cap} \underline{R}_k B$;
(3) $\overline{R}_k(\sim A) = \sim^* \overline{R}_k A$, $\underline{R}_k(\sim A) = \sim^* \underline{R}_k A$.

The grade upper (lower) outer (inner) boundary regions have been found to be the logical factors that hinder the equality relations when the grade approximation operators act on the union and intersection operations of sets. Definition 3, Definition 4 and Proposition 5 have extended and enriched rough set theory. In the operator-oriented view, it has added 4 operators: grade upper (lower) outer (inner) boundary operators. Thus, the old system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R})$ of the rough set theory is transformed into a new system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R}, bnR_k\overline{I}, bnR_k\underline{I}, bnR_k\overline{O}, bnR_k\underline{O})$. In the set-oriented view, it has added new set operations (i.e., $\underline{\cup}$, $\overline{\cap}$, $\sim^*$) to the classical set operations. Furthermore, the grade approximation operators have much better properties: they can maintain union, intersection and complement operations of sets. If the space with the new set operations is called an approximate power set space, then, in other words, the grade approximation operators are epimorphic mappings from the power set space to the approximate power set space.

4. Variable precision rough set model and its properties

4.1. Classification study of variable precision rough set model

Definition 5. Let $\beta \in [0, 1]$. $\overline{R}_\beta A = \cup\{[x]_R : c([x]_R, A) < 1 - \beta\}$ and $\underline{R}_\beta A = \cup\{[x]_R : c([x]_R, A) \leq \beta\}$ are called precision $1 - \beta$ R upper and lower approximations of A, respectively. $posR_\beta A = \overline{R}_\beta A \cap \underline{R}_\beta A$, $negR_\beta A =\sim (\overline{R}_\beta A \cup \underline{R}_\beta A)$, $UbnR_\beta A = \overline{R}_\beta A - \underline{R}_\beta A$, $LbnR_\beta A = \underline{R}_\beta A - \overline{R}_\beta A$, $bnR_\beta A = UbnR_\beta A \cup LbnR_\beta A$ are called precision $1 - \beta$ R positive and negative regions, upper and lower boundary regions, and boundary region of A, respectively. The upper and lower approximations, positive and negative regions, upper and lower boundary regions, and boundary region are all called the rough set regions of the variable precision rough set model.

The rough set regions of the variable precision rough set model extend both the variable precision approximations and the corresponding notions of the classical rough set model, which is similar to what the rough set regions of the graded rough set model do in Section 3.1. Similarly, they have their own logical meanings related to the precision quantitative index, which can be presented easily.

Theorem 3

(1) *When $\beta \in [0, 0.5)$, $\underline{R}_\beta A \subseteq \overline{R}_\beta A$, and $posR_\beta A = \underline{R}_\beta A$, $negR_\beta A =\sim \overline{R}_\beta A$, $LbnR_\beta A = \phi$, $UbnR_\beta A = bnR_\beta A = \overline{R}_\beta A - \underline{R}_\beta A$;*
(2) *When $\beta \in [0.5, 1)$, $\overline{R}_\beta A \subseteq \underline{R}_\beta A$, and $posR_\beta A = \overline{R}_\beta A$, $negR_\beta A =\sim \underline{R}_\beta A$, $UbnR_\beta A = \phi$, $LbnR_\beta A = bnR_\beta A = \underline{R}_\beta A - \overline{R}_\beta A$;*
(3) *When $\beta = 1$, $\overline{R}_\beta A = \phi$, $\underline{R}_\beta A = U$, and $posR_\beta A = \phi$, $negR_\beta A = \phi$, $UbnR_\beta A = \phi$, $LbnR_\beta A = U$, $bnR_\beta A = U$.*

Proof

(1) When $\beta \in [0, 0.5)$, i.e., $1 - \beta \in (0.5, 1]$, then $\beta < 1 - \beta$. Hence, $c([x]_R, A) \leq \beta \Rightarrow c([x]_R, A) < 1 - \beta$. From the definitions, $[x]_R \subseteq \underline{R}_\beta A \Rightarrow [x]_R \subseteq \overline{R}_\beta A$, so $\underline{R}_\beta A \subseteq \overline{R}_\beta A$.
(2) When $\beta \in [0.5, 1)$, i.e. $1 - \beta \in (0, 0.5]$, then $1 - \beta \leq \beta$. Hence, $c([x]_R, A) < 1 - \beta \Rightarrow c([x]_R, A) \leq \beta$. From the definitions, $[x]_R \subseteq \overline{R}_\beta A \Rightarrow [x]_R \subseteq \underline{R}_\beta A$, so $\overline{R}_\beta A \subseteq \underline{R}_\beta A$.
(3) When $\beta = 1$, i.e., $1 - \beta = 0$, $c([x]_R, A) \leq \beta$ always holds (because $c([x]_R, A) \in [0, 1]$), while $c([x]_R, A) < 1 - \beta$ never holds. From the definitions, $[x]_R \subseteq \underline{R}_\beta A$ always holds, while $[x]_R \subseteq \overline{R}_\beta A$ never holds. Hence, $\underline{R}_\beta A = U$ and $\overline{R}_\beta A = \phi$. $\square$

Theorem 3 is parallel to but different from the related results in Section 3.1, such as Proposition 2 and Theorem 1. According to Theorem 3, the rough set regions of the variable precision rough set model are relatively stable according to different classification cases of precision, and there are only 3 cases. Obviously, both the classical rough set model (where $\beta = 0$) and traditional variable precision rough set model (where $\beta \in [0, 0.5)$) belong to case (1).

Example 3. As for the medical example (Example 1), when $\beta = 0.4$, the following results are obtained: $\overline{R}_\beta A = [x]_2 \cup [x]_4 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_\beta A = posR_\beta A = [x]_4 \cup [x]_6 \cup [x]_8 \cup [x]_9$, $negR_\beta A = [x]_1 \cup [x]_3 \cup [x]_5$, $UbnR_\beta A = bnR_\beta A = [x]_2 \cup [x]_7$, $LbnR_\beta A = \phi$; $\underline{R}_\beta A \subseteq \overline{R}_\beta A$.

4.2. Properties of variable precision rough set model

The variable precision rough set model also extends the classical rough set model. Correspondingly, it lacks similar properties: (1) $\underline{R}A \subseteq A \subseteq \overline{R}A$, (2) $\overline{R}(\overline{R}A) = \underline{R}(\overline{R}A) = \overline{R}A$, $\overline{R}(\underline{R}A) = \underline{R}(\underline{R}A) = \underline{R}A$, and the equality formulas are also transformed into $\overline{R}_\beta(A \cup B) \supseteq \overline{R}_\beta A \cup \overline{R}_\beta B$ and $\underline{R}_\beta(A \cap B) \subseteq \underline{R}_\beta A \cap \underline{R}_\beta B$. Now, new properties in the 3 aspects will be studied in the variable precision rough set model, which is completely parallel to what the graded rough set model does in Section 3.3.

$\overline{R}_\beta A$ and $\underline{R}_\beta A$ have an inclusion relation based on Theorem 3. However, the variable precision approximations and the basic set do not have inclusion relations or equality relations. Evaluation mappings of sets may also be used to make similar estimates.

Theorem 4

(1) *When $\beta = 1$, $\overline{R}_\beta(\overline{R}_\beta A) = \overline{R}_\beta A = \phi$, $\underline{R}_\beta(\overline{R}_\beta A) = U$, $\overline{R}_\beta(\underline{R}_\beta A) = \phi$, $\underline{R}_\beta(\underline{R}_\beta A) = \underline{R}_\beta A = U$;*
(2) *When $\beta \in [0, 1)$, (i) $\overline{R}_\beta(\overline{R}_\beta A) = \overline{R}_\beta A = \underline{R}_\beta(\overline{R}_\beta A)$, (ii) $\overline{R}_\beta(\underline{R}_\beta A) = \underline{R}_\beta A = \underline{R}_\beta(\underline{R}_\beta A)$.*

Proof

(1) When $\beta = 1, \forall A \subseteq U$, then $\overline{R}_\beta A = \phi$ and $\underline{R}_\beta A = U$. Hence, the proof is easy.

(2) (i) If $[x]_R \subseteq \overline{R}_\beta(\overline{R}_\beta A)$, then $c([x]_R, \overline{R}_\beta A) < 1 - \beta$. If $[x]_R \not\subseteq \overline{R}_\beta A$, then $|[x]_R \cap \overline{R}_\beta A| = 0$. Hence, $c([x]_R, \overline{R}_\beta A) = 1$, and this contradicts $c([x]_R, \overline{R}_\beta A) < 1 - \beta$. Hence, $[x]_R \subseteq \overline{R}_\beta A$, and $\overline{R}_\beta(\overline{R}_\beta A) \subseteq \overline{R}_\beta A$. If $[x]_R \subseteq \underline{R}_\beta(\overline{R}_\beta A)$, then $c([x]_R, \overline{R}_\beta A) \le \beta$. If $[x]_R \not\subseteq \overline{R}_\beta A$, then $|[x]_R \cap \overline{R}_\beta A| = 0$. Hence, $c([x]_R, \overline{R}_\beta A) = 1$, and this contradicts $c([x]_R, \overline{R}_\beta A) \le \beta$ (here $\beta \in [0, 1)$). Hence, $[x]_R \subseteq \overline{R}_\beta A$, and $\underline{R}_\beta(\overline{R}_\beta A) \subseteq \overline{R}_\beta A$.

On the contrary, if $[x]_R \subseteq \overline{R}_\beta A$, then $[x]_R \cap \overline{R}_\beta A = [x]_R$, so $c([x]_R, \overline{R}_\beta A) = 0$. Hence, $c([x]_R, \overline{R}_\beta A) \le \beta$ and $c([x]_R, \overline{R}_\beta A) < 1 - \beta$. Hence, $[x]_R \subseteq \underline{R}_\beta(\overline{R}_\beta A)$ and $[x]_R \subseteq \overline{R}_\beta(\overline{R}_\beta A)$. Hence, $\overline{R}_\beta A \subseteq \underline{R}_\beta(\overline{R}_\beta A)$ and $\overline{R}_\beta A \subseteq \overline{R}_\beta(\overline{R}_\beta A)$.

(ii) If $[x]_R \subseteq \overline{R}_\beta(\underline{R}_\beta A)$, then $c([x]_R, \underline{R}_\beta A) < 1 - \beta$. If $[x]_R \not\subseteq \underline{R}_\beta A$, then $|[x]_R \cap \underline{R}_\beta A| = 0$. Hence, $c([x]_R, \underline{R}_\beta A) = 1$, and this contradicts $c([x]_R, \underline{R}_\beta A) < 1 - \beta$. Hence, $[x]_R \subseteq \underline{R}_\beta A$, and $\overline{R}_\beta(\underline{R}_\beta A) \subseteq \underline{R}_\beta A$. If $[x]_R \subseteq \underline{R}_\beta(\underline{R}_\beta A)$, then $c([x]_R, \underline{R}_\beta A) \le \beta$. If $[x]_R \not\subseteq \underline{R}_\beta A$, then $|[x]_R \cap \underline{R}_\beta A| = 0$. Hence, $c([x]_R, \underline{R}_\beta A) = 1$, and this contradicts $c([x]_R, \underline{R}_\beta A) \le \beta$ (here $\beta \in [0, 1)$). Hence, $[x]_R \subseteq \underline{R}_\beta A$, and $\underline{R}_\beta(\underline{R}_\beta A) \subseteq \underline{R}_\beta A$.

On the contrary, if $[x]_R \subseteq \underline{R}_\beta A$, then $[x]_R \cap \underline{R}_\beta A = [x]_R$, so $c([x]_R, \underline{R}_\beta A) = 0$. Hence, $c([x]_R, \underline{R}_\beta A) \le \beta$ and $c([x]_R, \underline{R}_\beta A) < 1 - \beta$ (here $\beta \in [0, 1)$). Hence, $[x]_R \subseteq \underline{R}_\beta(\underline{R}_\beta A)$ and $[x]_R \subseteq \overline{R}_\beta(\underline{R}_\beta A)$. Hence, $\underline{R}_\beta A \subseteq \underline{R}_\beta(\underline{R}_\beta A)$ and $\underline{R}_\beta A \subseteq \overline{R}_\beta(\underline{R}_\beta A)$. □

Theorem 4 is parallel to Theorem 2 in Section 3.3. According to Theorem 4, the variable precision approximation operators have trivial properties when $\beta = 1$ and idempotent properties in other cases. It is when $\beta = 1$ that hinders the idempotent properties of variable precision approximation operators.

Definition 6

$$bnR_\beta\overline{I}(A, B) = \cup\{[x]_R : [x]_R \not\subseteq \overline{R}_\beta A, \overline{R}_\beta B, [x]_R \subseteq \overline{R}_\beta(A \cup B)\},$$

$$bnR_\beta\underline{I}(A, B) = \cup\{[x]_R : [x]_R \not\subseteq \underline{R}_\beta A, \underline{R}_\beta B, [x]_R \subseteq \underline{R}_\beta(A \cup B)\},$$

$$bnR_\beta\overline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq \overline{R}_\beta A, \overline{R}_\beta B, [x]_R \not\subseteq \overline{R}_\beta(A \cap B)\},$$

$$bnR_\beta\underline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq \underline{R}_\beta A, \underline{R}_\beta B, [x]_R \not\subseteq \underline{R}_\beta(A \cap B)\}$$

are called the precision $1 - \beta$ R upper inner, lower inner, upper outer, and lower outer boundary regions of A and B, respectively. $bnR_\beta\overline{I}$, $bnR_\beta\underline{I}$, $bnR_\beta\overline{O}$ and $bnR_\beta\underline{O}$ are called the precision $1 - \beta$ upper inner, lower inner, upper outer, and lower outer boundary operators, respectively.

Definition 7. New union, intersection and complement operations of variable precision approximations (noted as $\underline{\cup}$, $\overline{\cap}$, $\sim^*$) are defined as follows:

(1) $\overline{R}_\beta A \underline{\cup} \overline{R}_\beta B = \overline{R}_\beta A \cup \overline{R}_\beta B \cup bnR_\beta\overline{I}(A, B)$,
$\underline{R}_\beta A \underline{\cup} \underline{R}_\beta B = \underline{R}_\beta A \cup \underline{R}_\beta B \cup bnR_\beta\underline{I}(A, B)$;
(2) $\overline{R}_\beta A \overline{\cap} \overline{R}_\beta B = \overline{R}_\beta A \cap \overline{R}_\beta B - bnR_\beta\overline{O}(A, B)$,
$\underline{R}_\beta A \overline{\cap} \underline{R}_\beta B = \underline{R}_\beta A \cap \underline{R}_\beta B - bnR_\beta\underline{O}(A, B)$;
(3) $\sim^* \overline{R}_\beta A = \sim \underline{R}_\beta A$, $\sim^* \underline{R}_\beta A = \sim \overline{R}_\beta A$.

Proposition 6

(1) $\overline{R}_\beta(A \cup B) = \overline{R}_\beta A \underline{\cup} \overline{R}_\beta B$, $\underline{R}_\beta(A \cup B) = \underline{R}_\beta A \underline{\cup} \underline{R}_\beta B$;
(2) $\overline{R}_\beta(A \cap B) = \overline{R}_\beta A \overline{\cap} \overline{R}_\beta B$, $\underline{R}_\beta(A \cap B) = \underline{R}_\beta A \overline{\cap} \underline{R}_\beta B$;
(3) $\overline{R}_\beta(\sim A) = \sim^* \overline{R}_\beta A$, $\underline{R}_\beta(\sim A) = \sim^* \underline{R}_\beta A$.

The studies of the variable precision rough set model are completely parallel to the previous ones performed for the graded rough set model (in Section 3.3). The precision upper (lower) outer (inner) boundary regions are just the logical factors that hinder equality relations of set operations. These results extend and enrich rough set theory. In the operator-oriented view, it adds 4 operators: precision upper (lower) outer (inner) boundary operators, and thus the old system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R})$ is transformed into a new system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R}, bnR_\beta\overline{I}, bnR_\beta\underline{I}, bnR_\beta\overline{O}, bnR_\beta\underline{O})$. In the set-oriented view, it adds new set operations to the classical set operations. Furthermore, the variable precision approximation operators are epimorphic mappings from the power set space to the approximate power set space.

5. Corresponding results in classical rough set model

The classical rough set model is a special case of the 2 models. Therefore, the corresponding results in the classical rough set model can be obtained according to the above results.

Corollary 1. $\underline{R}A \subseteq \overline{R}A$, $posRA = \underline{R}A$, $negRA =\sim \overline{R}A$, $bnRA = \overline{R}A - \underline{R}A$.

Example 4. The following results can be obtained from the medical example (Example 1): $\overline{R}A = [x]_2 \cup [x]_3 \cup [x]_4 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}A = posRA = [x]_4 \cup [x]_9$, $bnRA = [x]_2 \cup [x]_3 \cup [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8$; $\underline{R}A \subseteq \overline{R}A$.

Corollary 2

(1) $\overline{R}\phi = \underline{R}\phi = \phi$, $\overline{R}U = \underline{R}U = U$; (2) $\underline{R}A \subseteq A \subseteq \overline{R}A$; (3) $A \subseteq B \Rightarrow \overline{R}A \subseteq \overline{R}B$, $A \subseteq B \Rightarrow \underline{R}A \subseteq \underline{R}B$; (4) $\overline{R}(A\cup B) = \overline{R}A \cup \overline{R}B$, $\underline{R}(A \cup B) \supseteq \underline{R}A \cup \underline{R}B$; (5) $\overline{R}(A\cap B) \subseteq \overline{R}A \cap \overline{R}B$, $\underline{R}(A \cap B) = \underline{R}A \cap \underline{R}B$; (6) $\overline{R}(\sim A) =\sim \underline{R}A$, $\underline{R}(\sim A) =\sim \overline{R}A$; (7) $\overline{R}(\overline{R}A) = \underline{R}(\overline{R}A) = \overline{R}A$, $\overline{R}(\underline{R}A) = \underline{R}(\underline{R}A) = \underline{R}A$.

Corollary 3

(1) $bnR_0\overline{I}(A, B) = \phi$;
(2) $bnR_0\underline{I}(A, B) = \cup\{[x]_R : [x]_R \subseteq bnRA \cap bnRB, [x]_R \subseteq A \cup B\}$;
(3) $bnR_0\overline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq bnRA \cap bnRB, [x]_R \cap (A - B) \neq \phi, [x]_R \cap (B - A) \neq \phi\}$;
(4) $bnR_0\underline{O}(A, B) = \phi$.

Proof

(1) and (4) are easy to prove.
(2) If $[x]_R \subseteq \underline{R}(A \cup B)$, then $[x]_R \subseteq A \cup B$. (i) If $[x]_R \subseteq A$, then $[x]_R \subseteq \underline{R}A$; (ii) If $[x]_R \subseteq\sim A$, then $[x]_R \subseteq B$ and $[x]_R \subseteq \underline{R}B$; (iii) If $[x]_R \cap A \neq \phi$ and $[x]_R \cap (\sim A) \neq \phi$, then $[x]_R \subseteq bnRA$. $[x]_R \cap B \neq \phi$, so $[x]_R \subseteq bnRA$ and $[x]_R \subseteq \overline{R}B$. If $[x]_R \subseteq B$, then $[x]_R \subseteq \underline{R}B$. Otherwise, $[x]_R \subseteq bnRA$ and $[x]_R \subseteq bnRB$, and $[x]_R \subseteq A \cup B$.
Hence, $bnR_0\underline{I}(A, B) = \cup\{[x]_R : [x]_R \subseteq bnRA \cap bnRB, [x]_R \subseteq A \cup B\}$ from the definition and the completeness.
(3) If $[x]_R \subseteq \overline{R}A \cap \overline{R}B$, then $[x]_R \cap A \neq \phi$ and $[x]_R \cap B \neq \phi$. (i) If $[x]_R \cap (A \cap B) \neq \phi$, then $[x]_R \subseteq \overline{R}(A \cap B)$; (ii) Otherwise, $[x]_R \cap (A - B) \neq \phi$ and $[x]_R \cap (B - A) \neq \phi$, and then $[x]_R \subseteq bnRA$ and $[x]_R \subseteq bnRB$.

Hence, $bnR_0\overline{O}(A, B) = \cup\{[x]_R : [x]_R \subseteq bnRA \cap bn_RB, [x]_R \cap (A - B) \neq \phi, [x]_R \cap (B - A) \neq \phi\}$ from the definition and the completeness. □

Corollary 4

(1) $\overline{R}(A \cup B) = \overline{R}A\underline{\cup}\overline{R}B$, $\underline{R}(A \cup B) = \underline{R}A\underline{\cup}\,\underline{R}B$;
(2) $\overline{R}(A \cap B) = \overline{R}A\overline{\cap}\,\overline{R}B$, $\underline{R}(A \cap B) = \underline{R}A\overline{\cap}\underline{R}B$;
(3) $\overline{R}(\sim A) =\sim^* \overline{R}A$, $\underline{R}(\sim A) =\sim^* \underline{R}A$.

Similarly, the classical rough set theory is extended and enriched in both the operator-oriented and set-oriented views. It adds 2 operators: $bnR_0\underline{I}$ and $bnR_0\overline{O}$, and thus the old system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R})$ is transformed into a new system $(2^U, \cup, \cap, \sim, \overline{R}, \underline{R}, bnR_0\underline{I}, bnR_0\overline{O})$. On the other hand, it develops the following set operations: $\underline{\cup}$, $\overline{\cap}$, $\sim^*$, and the approximation operators are epimorphic mappings between the spaces. Furthermore, if Γ is a topology, then $\overline{R}\Gamma = \{\overline{R}G : G \in \Gamma\}$ and $\underline{R}\Gamma = \{\underline{R}G : G \in \Gamma\}$ satisfy the topology axiom with respect to the new set operations (i.e., $\underline{\cup}$ and $\overline{\cap}$). Therefore, the rough topology and approximate topology, innovative notions, can be defined and studied further.

6. Relationship and transformation between the 2 models

The variable precision rough set model and graded rough set model extend the classical rough set model according to precision and grade, respectively. The error degree level β and grade k are both parameters and confidence levels and are related to the relative and absolute errors, respectively. If β is given, then the structure and property of the model are determined. However, if k is given, the case in which the equivalence classes intersect the basic set must be discussed. Moreover, the properties of the 2 models are very similar. Therefore, there is a close relationship between them, and furthermore, they can be transformed mutually.

Definition 8. $\forall k, \forall \beta \in [0, 1], \forall [x]_R \in U/R$, $\beta([x]_R, k) = k/|[x]_R|$ is called "grade k error degree level" of $[x]_R$, and $k([x]_R, \beta) = \beta|[x]_R|$ is called "precision $1 - \beta$ grade" of $[x]_R$.

In terms of the definition of $c([x]_R, A)$, the following may be obtained:

$$c([x]_R, A) < 1 - \beta \Leftrightarrow |[x]_R \cap A| > k([x]_R, \beta),$$
$$c([x]_R, A) \leq \beta \Leftrightarrow |[x]_R \cap A| \geq |[x]_R| - k([x]_R, \beta).$$

Proposition 7

$$\overline{R}_\beta A = \cup\{[x]_R : |[x]_R \cap A| > k([x]_R, \beta)\},$$
$$\underline{R}_\beta A = \cup\{[x]_R : |[x]_R| - |[x]_R \cap A| \leq k([x]_R, \beta)\},$$
$$\overline{R}_k A = \cup\{[x]_R : c([x]_R, A) < 1 - \beta([x]_R, k)\},$$
$$\underline{R}_k A = \cup\{[x]_R : c([x]_R, A) \leq \beta([x]_R, k)\}.$$

In the variable precision rough set model and graded rough set model, both parameter β and parameter k are strict rules, and they act on all equivalence classes. According to the above result, the 2 models can make a mutual transformation based on both "precision grade"and "grade error degree level", but the transformation has a local attribute and a specific property. One model can be transformed into the other, but different equivalence classes should use distinctive parameters and establish concrete confidence levels in the transformation process.

The "grade error degree level" and "precision grade" are similar to the "error degree level" and "grade", respectively, except for the number field and number range. The "error degree level", " grade", "grade error degree level" and " precision grade" may be integers, rational numbers or real numbers for continuity. If $|[x]_R| < k$, then $\beta([x]_R, k) = k/|[x]_R| > 1$. This case exceeds the range of the error degree level. Hence, in a sense, the graded approximations are wider than the variable precision approximations. However, the variable precision approximations extend the graded approximations regardless of the range because $k([x]_R, \beta)$ may not be an integer.

This result proves that $|[x]_R| \leq k$ is a special case. In fact, if $|[x]_R| \leq k$, then $(\forall A \subseteq U)\ [x]_R \subseteq \underline{R}_k A$, while $[x]_R \not\subseteq \overline{R}_k A$ from the definitions. In other words, $|[x]_R| \leq k$ is a sufficient condition of $[x]_R \subseteq \underline{R}_k A$, while $|[x]_R| > k$ is a necessary condition of $[x]_R \subseteq \overline{R}_k A$.

Proposition 8

(1) *Suppose* $\underline{k} = min(k([x]_R, \beta))$ *and* $\overline{k} = max(k([x]_R, \beta))$, *where* $[x]_R \in U/R$. *Then,* $\overline{R}_{\overline{k}} A \subseteq \overline{R}_\beta A \subseteq \overline{R}_{\underline{k}} A$, *and* $\underline{R}_{\underline{k}} A \subseteq \underline{R}_\beta A \subseteq \underline{R}_{\overline{k}} A$. *If* $\overline{k} = \underline{k} = k$, *then* $\overline{R}_\beta A = \overline{R}_k A$ *and* $\underline{R}_\beta A = \underline{R}_k A$;

(2) *Suppose* $\underline{\beta} = min(\beta([x]_R, k))$ *and* $\overline{\beta} = max(\beta([x]_R, k))$, *where* $[x]_R \in U/R$. *Then,* $\overline{R}_{\overline{\beta}} A \subseteq \overline{R}_k A \subseteq \overline{R}_{\underline{\beta}} A$, *and* $\underline{R}_{\underline{\beta}} A \subseteq \underline{R}_k A \subseteq \underline{R}_{\overline{\beta}} A$. *If* $\overline{\beta} = \underline{\beta} = \beta$, *then* $\overline{R}_k A = \overline{R}_\beta A$ *and* $\underline{R}_k A = \underline{R}_\beta A$.

7. Conclusion

This paper has provided the basic structure and precise descriptions of rough set regions and has obtained some new properties of both the variable precision rough set model and graded rough set model. Moreover, this work has shown the relationship and transformation between the 2 models. The classical rough set model has provided many corresponding results, and the rough set theory has been extended and enriched with respect to both the operator-oriented and set-oriented views. The comparative study of the variable precision rough set model and graded rough set model is a basic study of the 2 models, where the graded rough set model has both a more complicated structure and richer properties than those of the variable precision rough set model. It has both important theoretical value and wide application prospects with respect to the 2 quantitative indexes: precision and grade. The study of this paper is based on equivalence partition and approximate space. More generally, the cases of common binary relation or general approximate space may be similarly studied. Furthermore, widespread applications and meaningful combinations of the 2 models are worth exploring.

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